Chaotic scattering in deformed optical microlasing cavities

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We consider a common class of dielectric optical microlasing cavities with quadrupolar deformations and address the question of the maximally allowed amount of deformation for both high-Q operation and a high degree of directionality of light emission. Our approach is to compute the probability for light rays to be trapped in the cavity by examining chaotic scattering dynamics in the classical phase space. We develop a dynamical criterion for high-Q operation and introduce a measure to quantify the directionality of the light emission. Our results suggest that high-Q and directionality can be achieved simultaneously in a wide range of the deformation parameter.

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I. INTRODUCTION

Optical processes in microcavities occur in important applications such as the development of microdisk semiconductor lasers [1,2] and optical fiber communications [3], in which total internal reflections of light are exploited to achieve nearly perfect mirror reflectivity. Dielectric cavities (cylinders or spheres) are a common type of optical microcavities. In such a situation, ideally the surface of the cavity confines certain modes of electromagnetic field, such as the ''whispering gallery'' (WG) modes [4–8] in which light circulates almost tangent to the surface of the cavity via total internal reflections, suffering minimal losses caused by evanescent leakage and scattering due to surface roughness. If there are no deformations in the cavity geometry from the ideal shape of cylinders or spheres, in a practical sense light can be trapped in the cavity indefinitely, making the cavity an ideal device for high-Q operation. This is the principle based on which the world's smallest lasers are fabricated [5,7,8].

While a circular symmetry permits WG modes with high-Q values, it prevents the laser emission from having a good directionality. Asymmetric resonant cavities (ARCs) with smooth deformations from the circular symmetry are then suggested [9–13]. Such deformations can be quite large, ranging from 1–50% with respect to the corresponding circular geometry. Although the WG modes of a spherical or cylindrical cavity can be treated analytically and the effect of small deformations can be analyzed using the traditional wave-perturbation theory, it is difficult to study cavities with large deformations as the modes of highly deformed cavities are not perturbatively related to these of the circular cavities. A question is then whether high-Q modes exist in highly deformed cavities. The pioneering works in Refs. [9–13] have shown that, for dielectric materials with a low index of refraction ($n<2$, such as glass fibers or cylindrical dye jets, assuming that the surrounding medium has $n_0=1$), if the cavity surface remains convex, high-Q WG modes can still exist. This important result is obtained, surprisingly, by studying chaotic dynamics resulting from classical ray tracing. Specifically, by treating waves propagating in ARCs as light rays bouncing within the cavity, the problem of ARCs becomes that of classical billiards, a paradigm for studying Hamiltonian chaotic dynamics. It is demonstrated that the Q value and the directionality of such a cavity can be computed directly from properties of chaotic ray dynamics, such as the particle-decay law in the phase space, which are found to be in good agreement with experimental measurements [9–13]. More recently, it is demonstrated both experimentally and computationally that for high-index semiconductor materials (index of refraction $n>2$), the WG modes may not be relevant to the lasing properties of the cavities [13]. Instead, resonant modes of ''bow-tie'' shapes are found to be responsible for the improved performance of the lasers in the presence of large geometric deformations.

The scope of this paper is restricted to low-index dielectric lasing cavities. Our focus is thus on the dynamics of the WG modes. We ask the following question: in order to achieve high-Q values of the lasing operation while maintaining a high degree of directionality, what is the maximally allowed amount of deformation from a circular symmetry? The answer to this question is relevant to the practical design of microdisk semiconductor lasers, where it is desirable to know the upper bound of the allowed deformation. To address this question, we choose the well-studied example of the class of two-dimensional (cylindrical) resonators with quadrupolar deformations from the circular boundary, and investigate the ray dynamics in the resonator from the standpoint of chaotic scattering [14] by focusing on the decay property of trajectories in the phase space. In general, the phase space contains both regular Kol'mogorov-Arnol'd-Moser (KAM) tori (surfaces) and chaotic regions, so chaotic scattering is nonhyperbolic [15–18]. The WG modes correspond then to chaotic-scattering trajectories in the phase space. For a given amount of deformation, the average lifetime of these trajectories in the scattering region, which correspond to light rays trapped in the cavity, determines the Q value of the cavity. Since in nonhyperbolic chaotic scattering, trajectories escape according to the law of algebraic decay, the average lifetime is determined by the exponent of the algebraic decay. By numerically computing the decay exponent as a function of the deformation parameter, we can establish the upper bound for the allowed deformation for any given index of refraction. To quantify the directionality of the light emission, we introduce a measure based on the probability distribution of the exiting angles of typical scattering trajectories. Our main result is that for typical low-index cavities, say with $n \sim 1.5$, deformation as large as 25% can be allowed for high-Q lasing operation with a high degree of directionality. We stress that, since we focus on the WG modes corresponding to scattering trajectories exterior to major KAM tori, our result is applicable only to low-index resonant cavities.

In Sec. II, we explain our qualitative criterion for high-Q operation based on the algebraic decay law. In Sec. III, we study the dynamics of cavities with quadrupolar deformations and introduce a measure to quantify the directionality of the light emission. Concluding remarks are presented in Sec. IV.

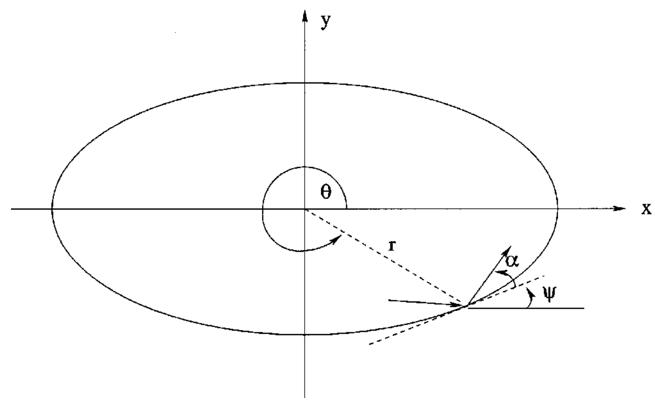

FIG. 1. Variables for constructing a Poincaré section for tracing the ray dynamics in a two-dimensional cavity.

II. DYNAMICAL CRITERION FOR HIGH-Q OPERATION

The quality factor, or the Q value, of a resonant cavity is given by $Q=\omega\tau$, where ω is the frequency of the resonant mode and τ is its lifetime in the cavity [9–13,19]. In the classical ray picture, τ is the average lifetime of phase-space trajectories in the scattering region, which correspond to the WG modes. In nonhyperbolic chaotic scattering, due to the "stickiness" effect of the KAM surfaces, the decay of trajectories is algebraic [20–25]. In particular, suppose a large number $N(0)$ of initial conditions, corresponding to various rays in the WG modes, is distributed in a phase-space region that does not contain any KAM tori, and let $N(t)$ be the number of trajectories still remaining in the cavity at time t. The survival probability $P(t)$ of trajectories in the cavity is approximately given by the ratio $N(t)/N(0)$. Because of chaos, initially $P(t)$ decays exponentially in time. But because of KAM surfaces, $P(t)$ decays algebraically after the initial exponential decay

$$P(t) \sim t^{-\beta}, \quad \text{for } t > t_0, \tag{1}$$

where t_0 (large) is the onset time of the algebraic decay and $\beta > 0$ is the algebraic decay exponent.

The stickiness effect and the resulting algebraic decay behavior [Eq. (1)] can be qualitatively understood, as follows. Consider the general case where chaotic regions and KAM surfaces coexist. The stickiness effect is that, if a particle is initialized in a chaotic region near some KAM surface, then the particle wanders close to that surface for a long time. Take, then, two nearby points on a given KAM surface and observe their evolution. What one typically finds is that the distance between the two points hardly changes with time, because the Lyapunov exponents in the directions along the KAM surface are zero (i.e., the motion is quasiperiodic). The symplectic nature of Hamiltonian dynamics implies that the Lyapunov spectrum is organized in pairs of exponents with equal value but opposite signs. Hence, an orbit on a KAM surface has zero Lyapunov exponents in directions both along and perpendicular to the surface. Due to ergodicity, a particle initialized in the chaotic region will come arbitrarily close to some KAM surface. When this occurs, the effective Lyapunov exponents will be small, leading to slow divergence of the particle trajectory from the KAM surface, thereby causing the stickiness effect.

The main consequence of the stickiness effect on the particle transport is that the particle-decay law becomes algebraic [20–25], in contrast to the exponential law observed in hyperbolic chaotic scattering or transient chaos in dissipative systems. The algebraic-decay law appears to hold not only for two-degrees-of-freedom Hamiltonian systems, but also for higher-dimensional systems such as those described by four- and six-dimensional symplectic maps [26]. Theoretical models [22,24,25] have also been proposed to explain the numerically observed algebraic-decay law. A fundamental assumption of these models is that a particle in the phase space executes a random walk between families of self-similar chains of KAM islands. Besides yielding the algebraic scaling law, these models also predict the values of the algebraic-decay exponent based on the number of self-similar families of islands included in the calculation. In light of these numerical and theoretical evidences, the algebraic decay law Eq. (1), as opposed to the more rapid exponential decay exhibited by hyperbolic systems, is thought to be characteristic of typical Hamiltonian systems.

Based on the algebraic-decay law [Eq. (1)], in the ray-dynamics picture of optical microcavities, the interplay between the amount of deformation and the algebraic-decay exponent can be seen, as follows. For a given value of the refraction index n and a circular symmetry, the operation in a WG mode stipulates that the angle α of incidence (see Fig. 1) satisfies: $\sin\alpha < \sin\alpha_c$, where α_c is the critical angle. For cavities with small deformations, the range of the angle α is small as the corresponding light ray circulates near in the boundary in the WG mode and, hence, it is more likely for the condition $\sin\alpha < \sin\alpha_c$ to be satisfied, leading to a high probability of light rays being trapped inside the cavity. We thus expect β to be small in such cases. Large deformations from the circular symmetry give rise to a large range of the angle α and, consequently, it is relatively easy for the condition of total internal reflection to be violated, leading to large values of β. Let ϵ be the parameter that characterizes the amount of deformation. We thus expect the algebraic decay exponent β to be a nondecreasing function of ϵ. To derive a criterion to determine the maximally allowed amount of deformation, we note that, heuristically, the average lifetime τ of scattering trajectories is given by

$$\tau \sim t_0 + \int_{t_0}^{\infty} tP(t)dt$$
$$= t_0 + t^{2-\beta}/(2-\beta)|_{t_0}^{\infty} \sim \begin{cases} C, & \text{if } \beta > 2 \\ \infty, & \text{if } \beta < 2, \end{cases} \quad (2)$$

where t_0 is the time of the onset of the algebraic decay and C is a constant. We see that if $\beta < 2$, then the average lifetime diverges asymptotically, indicating that in any practically long time scales, a high value of Q can be expected. If, in the corresponding range of the amount of deformation, a high directionality can be maintained (to be addressed below in numerical experiments), then a practical criterion for determining the upper bound of ϵ can be conveniently set as $\epsilon < \epsilon_c$, where $\beta(\epsilon_c) = 2$.

III. NUMERICAL EXAMPLE

We consider the following class of two-dimensional cavities, written in the polar coordinate (r,θ), with quadrupolar deformation characterized by ϵ

$$r(\theta) = \frac{1+\epsilon\cos 2\theta}{\sqrt{1+\epsilon^2/2}}. \quad (3)$$

Classical ray tracing can be done conveniently by using the Poincaré map defined with respect to the angles [27] ψ, α, and θ, as shown in Fig. 1. The map can be written in the following implicit form, relating the dynamical variables (θ,α) at successive total internal reflections off the boundary of the cavity:

$$\tan(\psi_t + \alpha_t) = \frac{(1+\epsilon\cos 2\theta_{t+1})\sin\theta_{t+1} - (1+\epsilon\cos 2\theta_t)\sin\theta_t}{(1+\epsilon\cos 2\theta_{t+1})\cos\theta_{t+1} - (1+\epsilon\cos 2\theta_t)\cos\theta_t},$$

$$\tan\psi_{t+1} = \frac{2\epsilon\sin\theta_{t+1}\sin 2\theta_{t+1} - \cos\theta_{t+1}(1+\epsilon\cos 2\theta_{t+1})}{\sin\theta_{t+1}(1+\epsilon\cos 2\theta_{t+1}) + 2\epsilon\cos\theta_{t+1}\sin 2\theta_{t+1}},$$

$$\alpha_{t+1} = \psi_{t+1} - \psi_t - \alpha_t, \quad (4)$$

where t is the discrete-time index denoting the event of bounce of light ray off the cavity boundary. In our numerical experiments, we fix the critical angle of incidence to be $\sin\alpha_c = 0.735$ (arbitrarily), corresponding to low-index cavities with refraction index $n \approx 1.475$. Total internal reflection breaks down when α exceeds α_c, which is regarded as the escape of the corresponding light ray or scattering trajectory. Figure 2(a) shows, for $\epsilon = 0.1$, the phase-space structure of the map in Eq. (4), where 1000 initial conditions chosen uniformly from $\alpha_0 \in [0.05, \pi/4]$ are utilized, and the horizontal dashed line indicates the critical angle of incidence. We see that the phase space contains both KAM tori and chaotic regions. While the map in Eq. (4) describes the dynamics of particles in a closed billard, imposing the threshold line at α_c makes the system effectively open. Figure 2(b) shows, in the two-dimensional physical space (x,y), a typical scattering trajectory in a WG mode and its escape from the cavity after about 1000 bounces. Figure 2(c) shows the dwelling time T of light rays in the cavity as a function of the initial angle of incidence α_0, where if the trajectory lives on a KAM torus, the time is infinite (represented by $T = 10^4$), and if the ray is in a chaotic region connected to α_c, it will eventually escape but the time it stays in the cavity can be long. Since WG modes typically correspond to trajectories in the open chaotic region above the binding KAM tori, the Q-value of the cavity in the WG modes is determined by motions of rays in the chaotic region. The plot in Fig. 2(c) in fact represents a scattering function [15–18]. The emission of light rays in the WG modes apparently has a high degree of directionality, as shown in Fig. 2(d), a histogram of the emission angle θ_{out}, where θ_{out} is defined to be the angle of the refracted, exiting light ray with respect to the x axis. We see that the probability distribution of outgoing angles of scattering trajectories appears to be highly localized.

In the presence of deformation, the probability for a typical light ray to sustain total internal reflection or, equivalently, that for a trajctory to survive in the scattering region, decays algebraically with time, as shown in Figs. 3(a)–3(c), where the survival probability $P(t)$ is plotted versus time on a logarithmic scale for $\epsilon = 0.1$, 0.2, and 0.3, respectively. To obtain these plots, we use $N(0) = 10^4$ initial conditions distributed uniformly in the open chaotic region in Fig. 2(a), compute $N(t)$, the number of trajectories that are still in the cavity at time t, and approximate $P(t) \approx N(t)/N(0)$. The approximate, yet robust linear behaviors in Figs. 3(a)–3(c) indicate the expected behavior of the algebraic decay, and the value of the decay exponent β apparently increases as the deformation becomes more severe. Figure 4(a) shows how the algebraic exponent increases as the deformation parameter ϵ is increased. We observe a smooth, monotonically increasing behavior. In particular, for $\epsilon < \epsilon_c \approx 0.21$, the exponent remains below the critical value $\beta_c = 2$, indicating that for $0 < \epsilon < \epsilon_c$, the average lifetime of light rays in the cavity diverges and the Q value of the cavity is high in a statistical sense. In contrary, for $\epsilon > \epsilon_c$, the algebraic-decay exponent is above two, implying relatively low Q values. Thus, for the particular low-index ($n \approx 1.5$) cavity with a quadrupolar deformation in our numerical study, in order to achieve a high-Q operation, the amount of deformation should not exceed the value of about 0.2.

While high-Q operation of the cavity is desired, an equally important measure is the directionality of the light emission. To quantify this, we consider the worst case where light is emitted equally probably in all directions. The probability distribution is thus: $P(\theta_{out}) = 1/2\pi$ for $0 \leq \theta_{out} \leq 2\pi$. For the computed distribution as in Fig. 2(d), we normalize the height of the distribution to $1/2\pi$ and compute the total area under the distribution curves

$$A(\epsilon) = \int_0^{2\pi} P(\theta_{out}) d\theta_{out},$$

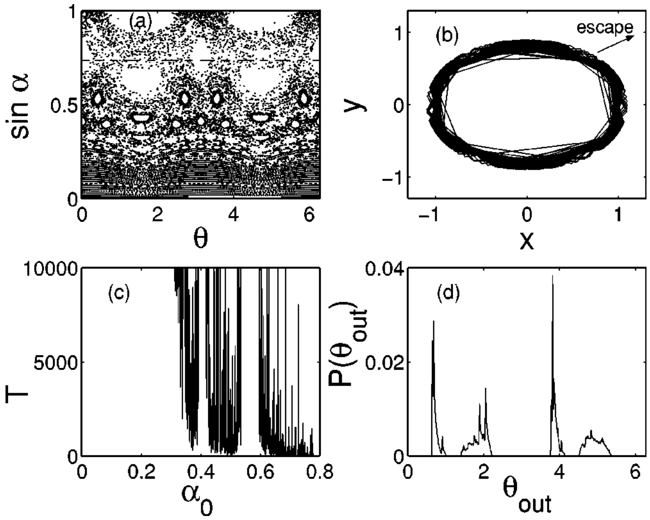

FIG. 2. For a dielectric cavity of refractive index $n = 1.475$ and a quadrupolar deformation of $\epsilon = 0.1$: (a) the phase-space structure, (b) behavior of light ray (in a WG mode) and its escape from the cavity, (c) staying time of light rays versus the initial angle of incidence, and (d) histogram of the emission angle θ_{out}. The highly localized pattern in the histogram indicates a high degree of directionality of the emitting light.

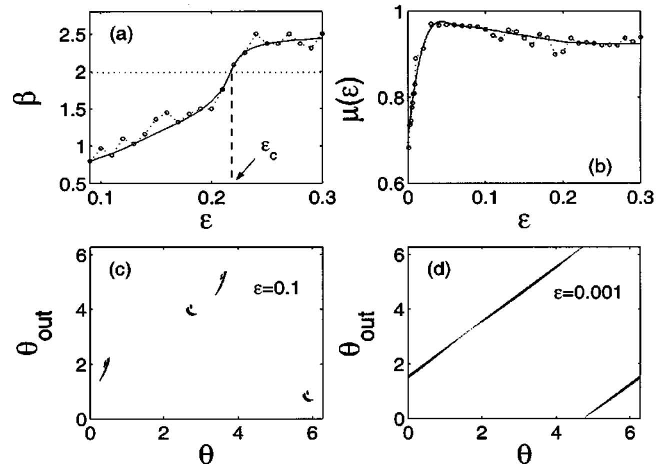

FIG. 4. For cavity of refractive index $n = 1.475$ and a quadrupolar deformation: (a) The algebraic-decay exponent β versus the deformation parameter ϵ. high-Q lasing operation can be expected for $\epsilon < \epsilon_c \approx 0.21$. (b) The measure μ of directionality versus ϵ. Apparently, a high degree of directionality can be maintained in the range of high-Q operation. (c,d) Position angle θ versus the emission angle θ_{out} for $\epsilon = 0.1$ and 0.001, respectively.

where a unit area indicates uniform emission. The following measure of directionality can then be defined:

$$\mu(\epsilon) = 1 - A(\epsilon), \tag{5}$$

where high values of μ signify high degrees of directionality (for uniform emission, $\mu = 0$). Figure 4(b) shows, for the particular cavity in Eq. (3), μ versus ϵ. Apparently, in the range of high-Q operation ($\epsilon < \epsilon_c$), $\mu \approx 1$, indicating a high degree of directionality. We also find that, in the range of the deformation parameter where light emissions possess a high degree of directionality, the rays appear to exit the cavity at only a few locations on the boundary. At each exiting point, the range of the emission angle θ_{out} is highly localized. This behavior is shown in Fig. 4(c), where the angle θ that defines the position of boundary point (position angle) versus the emission angle θ_{out} is plotted for the escaping light rays. In contrast, when the deformation is near zero, light rays can exit from almost everywhere on the boundary ($0 \leqslant \theta < 2\pi$), which means that, for the WG-mode operation, light can be emitted in almost every possible direction: $0 \leqslant \theta_{out} < 2\pi$, as shown in Fig. 4(d). Thus, for $\epsilon \gtrsim 0$, there exists apparently no particular direction for the light emission, as expected.

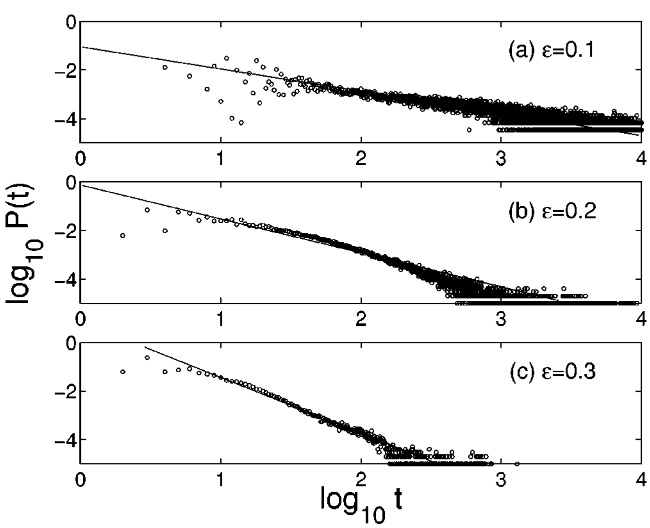

FIG. 3. For a dielectric cavity of refractive index $n \approx 1.475$ and a quadrupolar deformation, algebraic decay of the probability for light rays to be trapped in the cavity for (a) $\epsilon = 0.1$, (b) $\epsilon = 0.2$, and (c) $\epsilon = 0.3$.

IV. CONCLUSION

In summary, we have investigated the ray dynamics in optical microlasing cavities with quadrupolar deformation from the perspective of chaotic scattering. A physical argument, based on examining the survival probability of scattering trajectories in the cavity, is provided to assess the Q value of the cavity. We also introduce a measure to quantify the directionality of the light emission. These ideas are applicable to deformed optical cavities in general, although we use quadrupolar deformations in numerical computations. Our result that high-Q operation and high degrees of directionality can be achieved simultaneously in deformed cavities can be potentially useful for practical design and fabrication of these cavities, which are key to the development of microlasers.

We remark that in actual operation of a microlaser, the issue of dissipation may be important. Evanescent leakage and surface roughness are possible contributing factors to the dissipation. The influence of weak dissipation on chaotic scattering, particularly on nonhyperbolic scattering, can be substantial. For instance, the decay law for scattering trajectories can change metamorphically from being algebraic to being exponential in the presence of an arbitrarily small amount of dissipation [28]. Rigorously speaking, the algebraic decay law is not observable in optical cavities with dissipation. Nonetheless the algebraic law is useful for estimating the Q value of the cavity, which is important for the design of microlasers.

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[1] Y. Yamamoto and R.E. Slusher, Phys. Today **46**(6), 66 (1993).
[2] R.K. Chang and A.J. Campillo, *Optical Processes in Microcavities* (World Scientific Publishing, Singapore, 1996).
[3] J.F. Owen, P.W. Barber, P.B. Dorain, and R.K. Chang, Phys. Rev. Lett. **47**, 1075 (1981).
[4] S.-X. Qian, J. B. Snow, H.-M. Tzeng, and R.K. Chang, Science **231**, 486 (1986).
[5] S.L. McCall, A.F.J. Levi, R.E. Slusher, S.J. Pearton, and R.A. Logan, Appl. Phys. Lett. **60**, 289 (1991).
[6] A.F.J. Levi, R.E. Slusher, S.L. McCall, S.J. Pearton, and W.S. Hobson, Appl. Phys. Lett. **62**, 2021 (1993).
[7] H. Deng, Q. Deng, and D.G. Deppe, Appl. Phys. Lett. **69**, 3120 (1996).
[8] T. Baba, IEEE J. Sel. Top. Quantum Electron. **3**, 808 (1997).
[9] J.U. Nöckel, A.D. Stone, and R.K. Chang, Opt. Lett. **19**, 1693 (1994).
[10] A. Mekis, J.U. Nöckel, G. Chen, A.D. Stone, and R. K. Chang, Phys. Rev. Lett. **75**, 2682 (1995).
[11] J.U. Nöckel, A.D. Stone, G. Chen, H.L. Grossman, and R.K. Chang, Opt. Lett. **21**, 1609 (1996).
[12] J.U. Nöckel and A.D. Stone, Nature (London) **385**, 45 (1997).
[13] C. Gmachl, F. Capasso, E.E. Narimanov, J.U. Nöckel, A.D. Stone, J. Faist, and D.L. Sivco, Science **280**, 1556 (1998).
[14] T. Tél and E. Ott, Chaos **3**, 417 (1993).
[15] Y.-T. Lau, J.M. Finn, and E. Ott, Phys. Rev. Lett. **66**, 978 (1991).
[16] Y.-C. Lai, R. Blümel, E. Ott, and C. Grebogi, Phys. Rev. Lett. **68**, 3491 (1992).
[17] M. Ding, C. Grebogi, E. Ott, and J.A. Yorke, Phys. Rev. A **42**, 7025 (1990).
[18] Y.-C. Lai, M. Ding, C. Grebogi, and R. Blümel, Phys. Rev. A **45**, 4661 (1992).
[19] I. Triandaf and I.B. Schwartz, Phys. Rev. E **61**, 3601 (2000).
[20] C.F.F. Karney, Physica D **8**, 360 (1983).
[21] J.D. Meiss, J.R. Cary, C. Grebogi, J.D. Crawford, A.N. Kaufman, and H.D.I. Abarbanel, Physica D **6**, 375 (1983).
[22] J.D. Hanson, J. Cary, and J.D. Meiss, J. Stat. Phys. **39**, 327 (1985).
[23] B.V. Chirikov and D.L. Shepelyansky, Physica D **13**, 395 (1984).
[24] J.D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985).
[25] J.D. Meiss and E. Ott, Physica D **20**, 387 (1986).
[26] M. Ding, T. Bountis, and E. Ott, Phys. Lett. A **151**, 395 (1990).
[27] M.V. Berry, Eur. J. Phys. **2**, 91 (1981).
[28] A.E. Motter and Y.-C. Lai, Phys. Rev. E **65**, 015205(R) (2001).